A conservation law formulation of nonlinear elasticity in general relativity

Classical Quantum Grav. **29** 015005

C. Gundlach, I. Hawke, S. Erickson

School of Mathematics,
University of Southampton, UK

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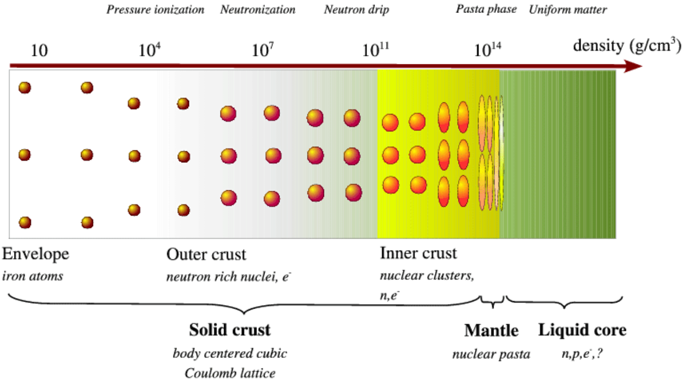
Outline

1. Neutron star crusts
   - Crustal properties
   - Crustal evolution
2. Continuum mechanics
   - Matter space
   - Dynamics
3. Numerical implementation
   - Equations
   - Evolution
4. Results
   - Simple Shock tubes
   - Multi-D
5. Going further
   - Coupling
   - Conclusions

Crust structure



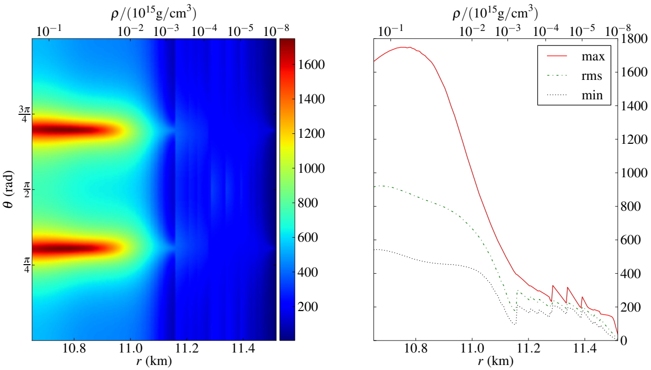
Chamel & Haensel, Liv. Rev. Relativity

For a cold NS the crust may extend from $\rho \sim 10^4$–10^{14}g cm^{-3}.

Impurities irrelevant; breaking strain large (Horowitz et al.).

A crystalline QCD core is an exotic possibility.

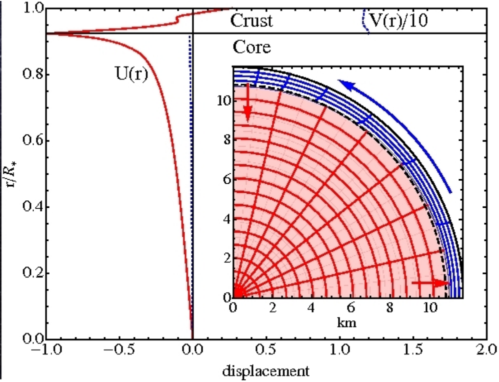
Crustal evolution



In binary inspiral, tidal effects will partially crack the crust only late on (Penner et al.).

However, resonant interface modes may shatter the whole crust (Tsang et al.).

Crustal evolution



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Matter space

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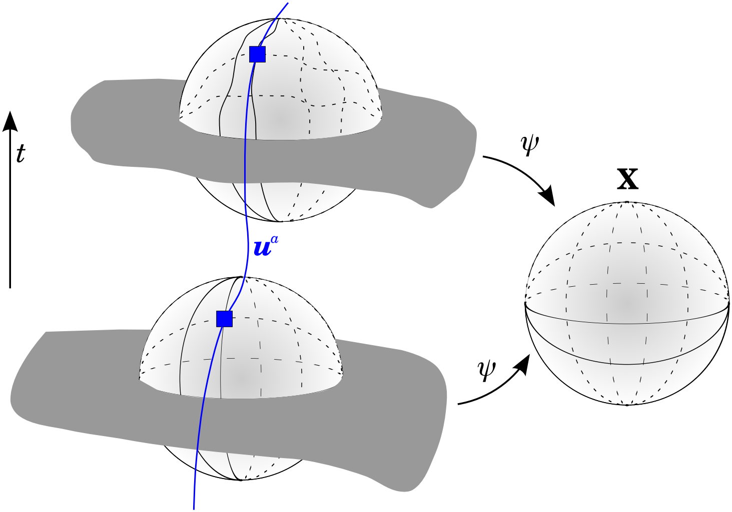
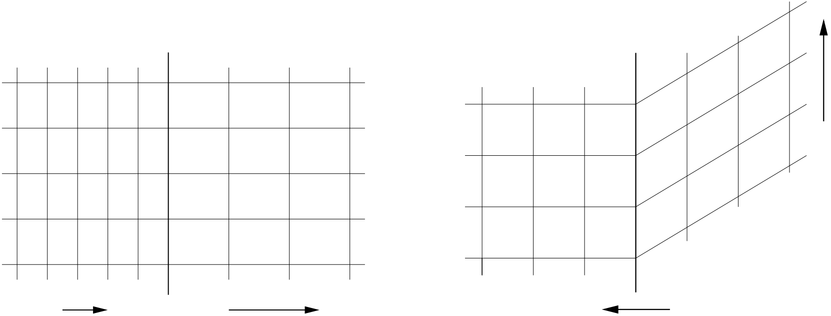


A *body* is given by a *reference configuration* X, and its deformation computed from the map ψ.

Shocks and the fluid limit



Standard fluid shocks are possible. Jump conditions $[\psi_y^{Y,X}] = 0$ forbid other discontinuities.

The fluid limit is singular. Integrability conditions $\psi^A{}_{[i,j]}$ connected to hyperbolicity questions.

Shocks and the fluid limit

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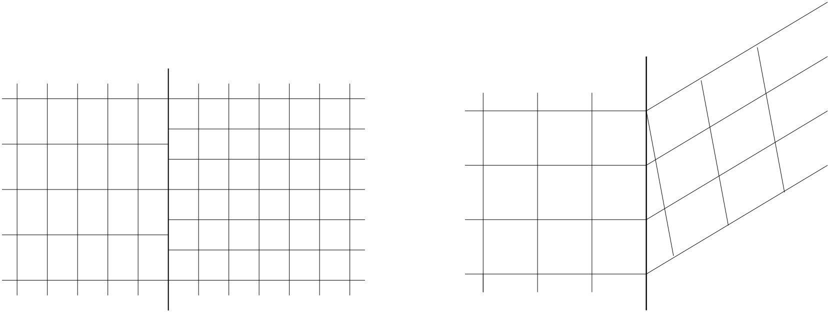

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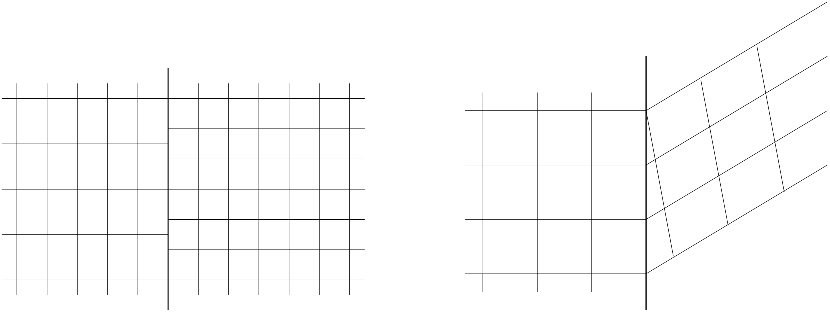

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Dynamics

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The stress-energy tensor is that of hydro, plus anisotropic terms π^{ab}:

$$T^{ab} = (e + p)u^a u^b + p\, g^{ab} + \pi^{ab}.$$

This gives the balance laws

$$\left(\sqrt{\gamma_x}\mathcal{U}\right)_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms},$$

with (introducing $\pi = v^i v^j \pi_{ij} = \gamma^{ij}\pi_{ij}$, and ignoring gauge terms)

$$\mathcal{U} = \begin{pmatrix} D \\ S_j \\ \tau \end{pmatrix} = \begin{pmatrix} nW \\ nhW^2 v_j + \pi_{ij} v^i \\ nhW^2 - p - D - \pi \end{pmatrix}, \quad \mathcal{F}^i \sim \begin{pmatrix} D\hat{v}^i \\ nhW^2 v_j \hat{v}^i + p\delta^i{}_j + \pi^i{}_j \\ (nhW^2 - D)\hat{v}^i + \pi^{0i} \end{pmatrix}.$$

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Equations

For completeness we note the full system:

$$k_{AB,t} + \hat{v}^j k_{AB,j} = 0,$$
$$\psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 2\hat{v}^j \psi^A{}_{[i,j]},$$

and, as given earlier

$$(\sqrt{\gamma_x}\mathcal{U})_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms.}$$

We also have constraints

$$\psi^A_{[i,j]} = 0,$$

and an EOS $\epsilon \equiv \epsilon(n, I^1, I^2, s)$ where $n, I^{1,2}$ are scalar invariants of $k^A{}_B$.

Con2Prim

Converting $(k_{AB}, \psi^A{}_i, S_j, \tau) \to (v^i, p)$ is the only remaining task.

Standard iterative approach:

1. *Guess* four quantities: $\overline{p - \pi}$ and $\overline{\pi_{ij} v^j}$;
2. Compute all terms consistent with the guess; in particular, $\overline{n}$, $\overline{I^{1,2}}$, $\overline{s}$ can be found;
3. Use the EOS to compute p and π_{ab} from the above;
4. Compute the residuals for the guesses.

Reduces to standard approach for hydro; *very* expensive (50% of computational time).

Outline

Newtonian shock tube

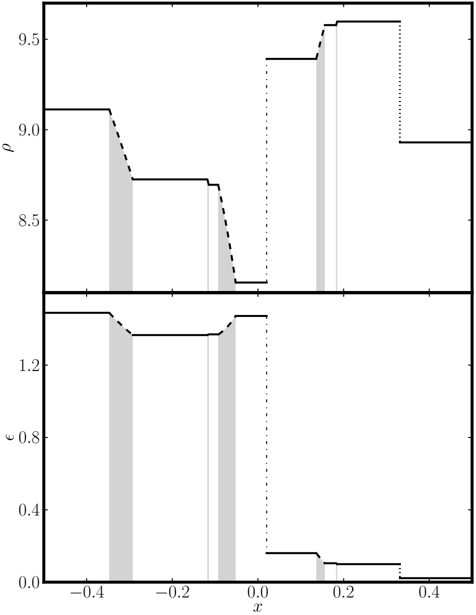
A Newtonian shock tube with all waves.

Only the right wave is a shock. Some rarefactions are very steep.

Results using 1000 points (100 shown).

All features well captured. No oscillations. Minor under/over shoots.

2- and 6-waves only clear in deformation components.



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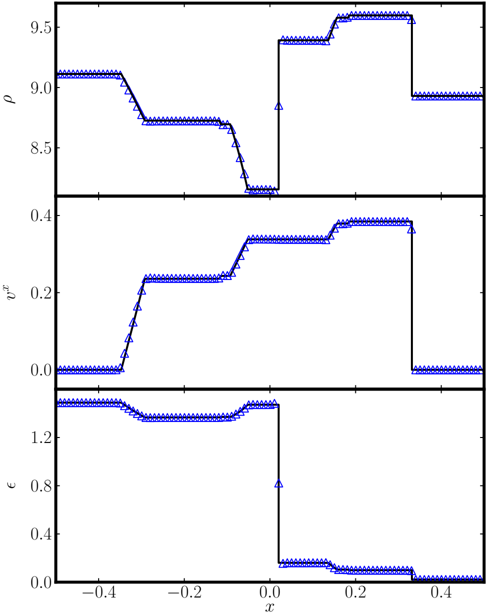

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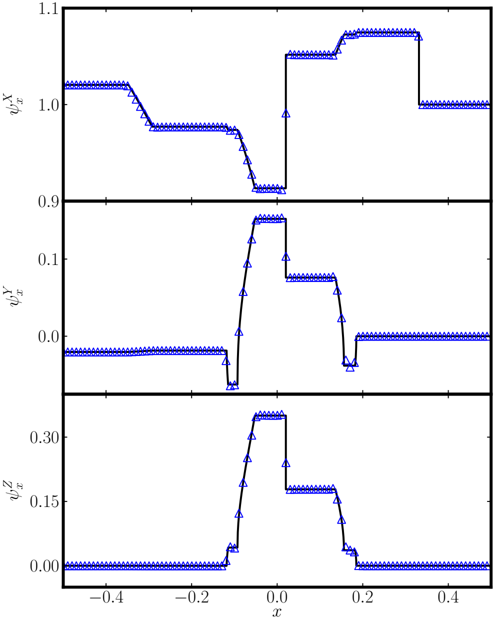

SR shock tube

A relativistic shock tube with 4 waves - no contact, 3- or 5-wave.

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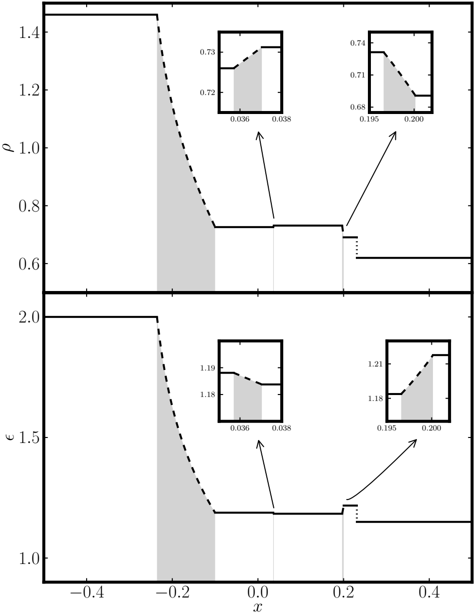
Strong deformation best seen in ψ_x^Y.



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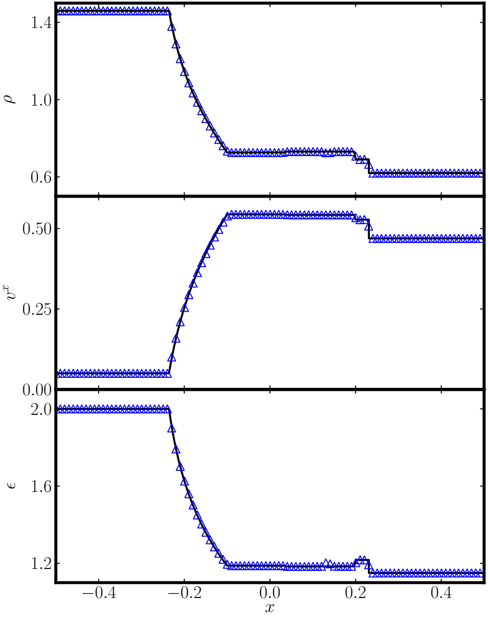

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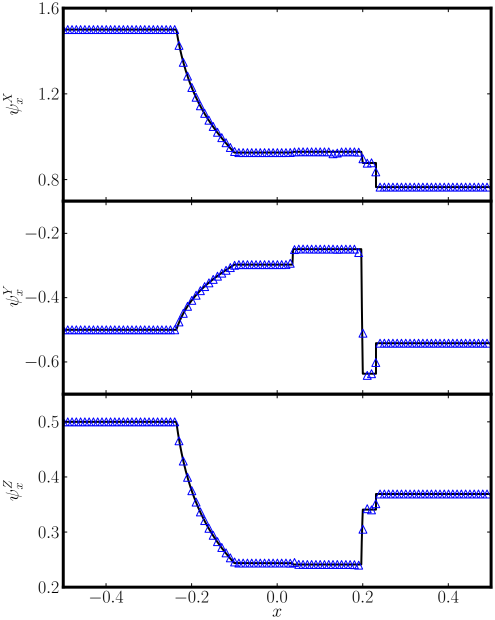
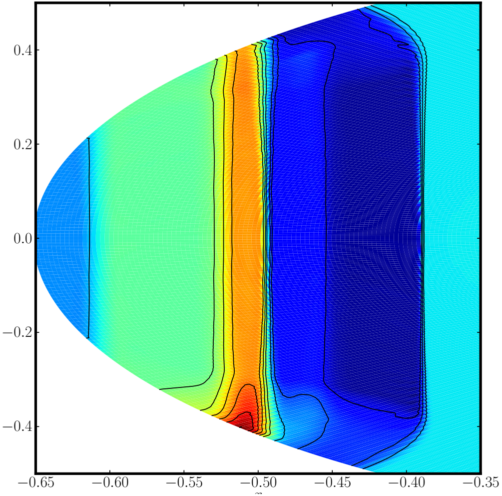


Complex shock tubes

Use cylindrical coordinates with the Newtonian shock.

Tests sources, non-trivial metric.

No problems. Expected secular drift from boundary conditions.



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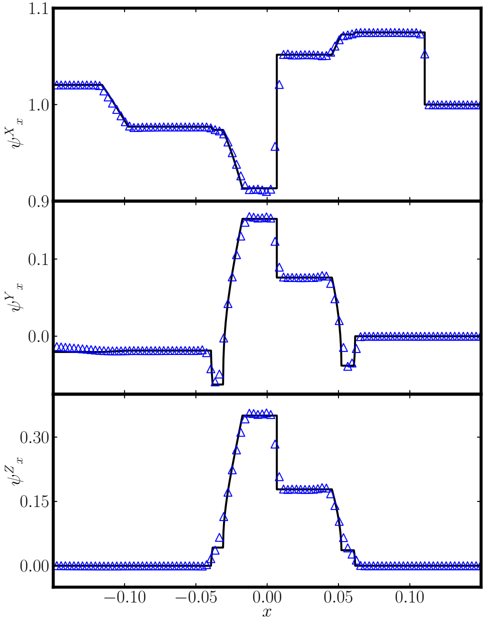

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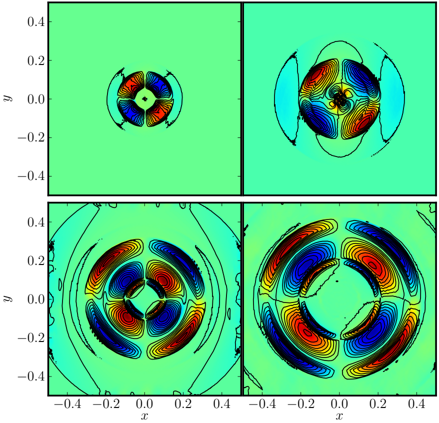
Rotor tests

Newtonian literature suggests problems with naive evolution of ψ:

1. hyperbolicity issues explain this;
2. fixes can be implemented
   1. constraint addition in sources stabilizes it
   2. constraint damping used by some groups.

However, no problem with rotor tests in Newtonian or SR!



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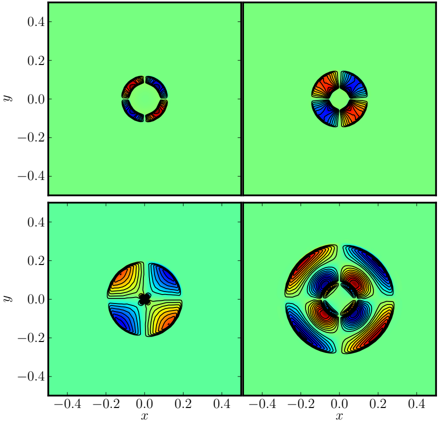

Outline

Coupling

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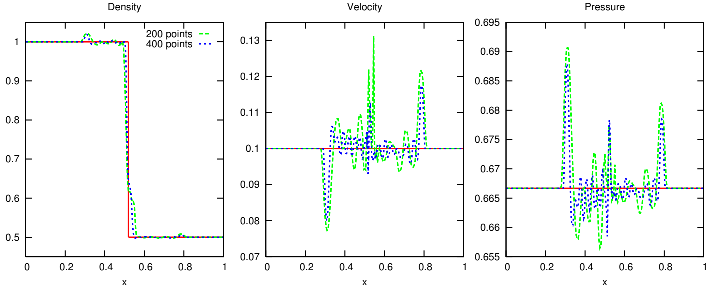
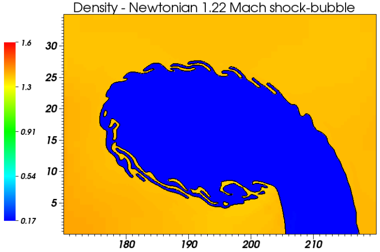


Attempts to model crust-core transition by "smearing" $\breve{\mu}$ fail. Need level sets (e.g. Millmore & Hawke) or similar.



Conclusions

- Elasticity alone a “straightforward” extension.
- ET implementation underway:
  - Basic shock tests work;
  - Use to test multi-D constraint issues with mesh refinement – already suggesting issues with hyperbolicity?
- Outstanding questions include
  1. Accurate numerics – characteristic structure really complex
  2. Multi-D issues – especially constraints
  3. Weak solution existence/uniqueness implies EOS constraints?
  4. Multi-material coupling, and melting/freezing.
  5. Shattering – fracture mechanics, wave propagation.
  6. Coupling to magnetic fields.

Implementation details

We note that by assumption k_{AB} is differentiable. We can thus evolve k_{AB} using naive central differencing.

We then choose primitive variables $(\psi^A{}_i, v^i, p)$ and evolve the remaining equations using standard HRSC methods:

- MoL – typically RK3;
- Slope limiting RSA – typically van Leer MC;
- HLL flux – typically with excessive dissipation.

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EOS

The EOS depends on the strain g^{AB} compared to the reference k_{AB} and e.g. the entropy, in addition to any polarizing effects.

Simplify in two ways:

1. **Homogeneous**: $\epsilon \equiv \epsilon(g^{AB}, k_{AB}, s)$
2. **Isotropic**: $\epsilon \equiv \epsilon(\rho, I^{1,2}, s)$ – the strain dependence is encoded in the invariants of $k^{A}{}_{B}$.

Simple tests here use toy EOS using gamma-law fluid plus term proportional to *a* shear scalar.

Existence and uniqueness of weak solutions requires EOS restrictions (as yet unclear).

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